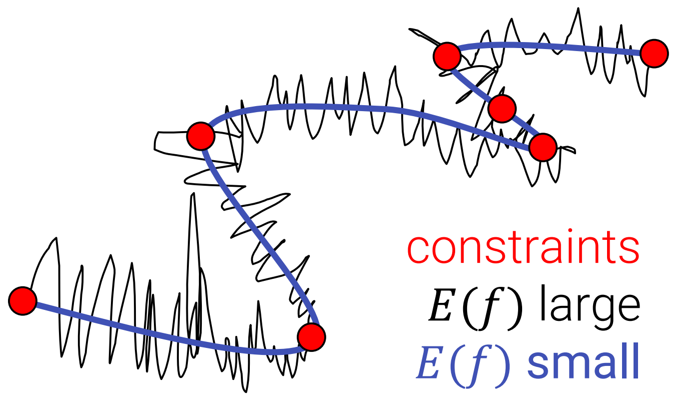
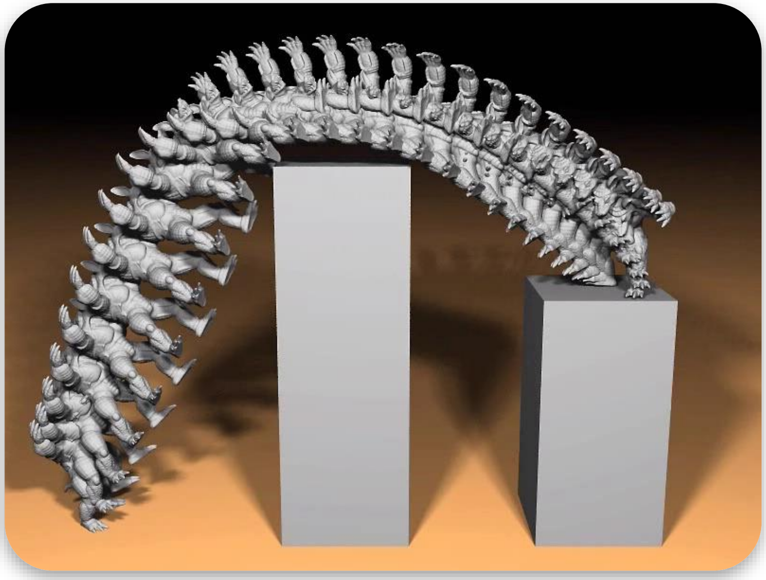
Modelling 1

SUMMER TERM 2020

JOHANNES GUTENBERG UNIVERSITÄT MAINZ





LECTURE 21

Variational Modeling

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Variational Modeling

Basic Techniques

Calculus of Variation

Basic Idea:

- Consider functions
$$f: S \to D$$
- Define an "*energy functional*"
$$E: (S \to D) \to \mathbb{R}$$
- *Functionals* map functions $(\cdot \to \cdot)$ to numbers $(\mathbb{R})$
- Interpretation: "score"
  - Usually: "energy"
  - I.e., the smaller the better

Calculus of Variation

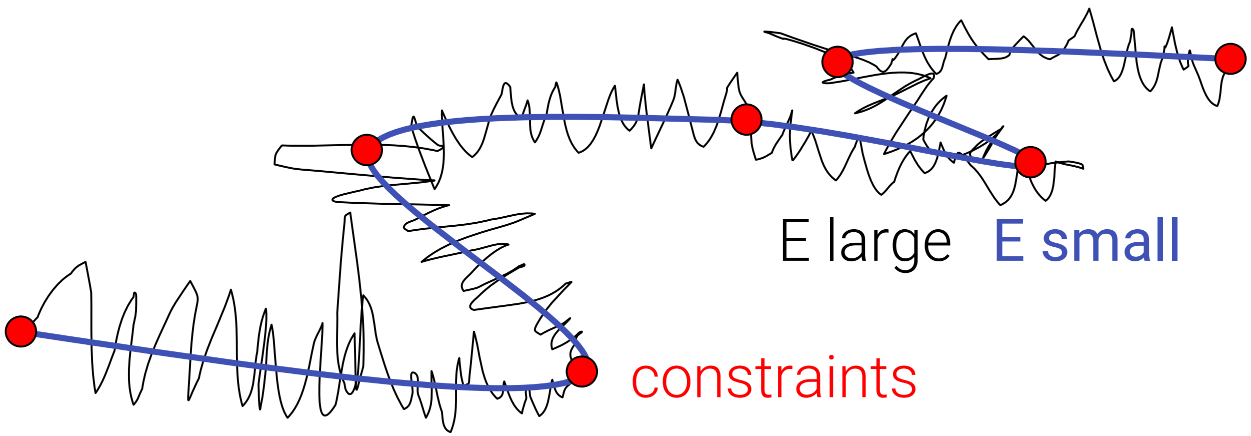
Building energy functionals

- Encode requirements ("constraints") on $f: S \to D$
  - *Soft constraints* $\to$ violation increases energy.
  - *Hard constraints* $\to$ violation not allowed
    - Excluded from S.

Solution by optimization

- Compute the function(s) f that minimize E.

Calculus of Variation



General framework

- Model problems by "wishlists"

Example 1

- We are looking for a curve.
- It should be as smooth as possible.
- Hard constraint: pass through a number of points

Calculus of Variation

Another example

- **Problem**
  - We want to go to the moon.
- **Given**
  - Orbits of moons, planets and star(s).
  - Flight conditions (atmosphere, gravitation of stellar bodies)
- **Unknowns**
  - Throttle from rocket motors (vector function $\mathbf{x}(t): \mathbb{R} \to \mathbb{R}^3$)
- **Energy function**
  - Usage of rocket fuel (the fewer the better)
  - Perhaps: Overall travel time (maybe not longer than a week)

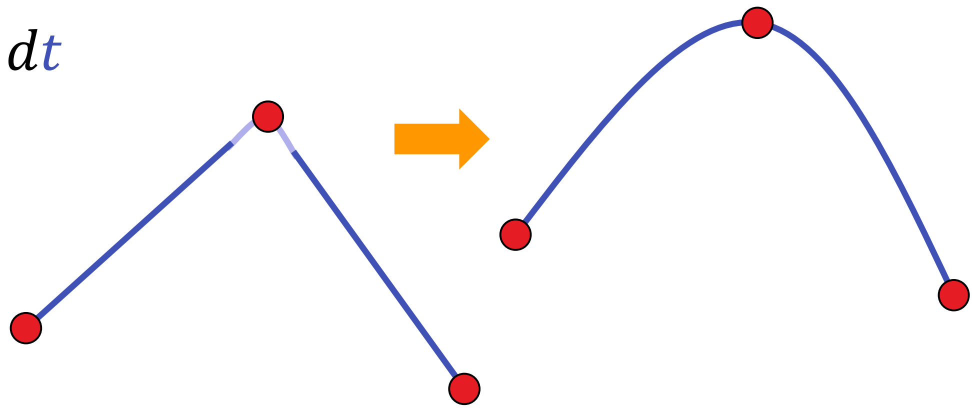
Calculus of Variation

To the moon

- **Constraints**
  - Start in Cape Canaveral (upright).
  - End up on the moon.
  - Do not hit moons or planets on the way.
  - Land on the moon at $\leq$ 20 km/h relative speed.
  - Rocket motor has a limited range of forces
    - Minimum and maximum power
    - Angle limitations
    - No backward thrust
- Flying to the moon = minimizing a functional
  - Very, very slightly simplified...

A Simple Example

$$E(f) = \int_{t=0}^{t=T} \left(\frac{d^2}{dt^2} f(t) \right)^2 dt$$



Simple example: variational splines

- We want smooth curves
  - Small curvature
  - Approximated by small second derivatives
    - (Correct curvature is nonlinear)
  - Quadratic energy

A Simple Example

Simple example: variational splines

- Soft constraints
  - Parameter values $t_1, \ldots, t_n$ at which we should approximate points $\mathbf{p}_1, \ldots, \mathbf{p}_n$:

$$E(f) = \int_{t=t_1}^{t=t_n} \left[\frac{d^2}{dt^2} f(t)\right]^2 dt + \lambda \sum_{i=1}^{n} (f(t_i) - \mathbf{p}_i)^2$$

  - λ controls smoothness

A Simple Example

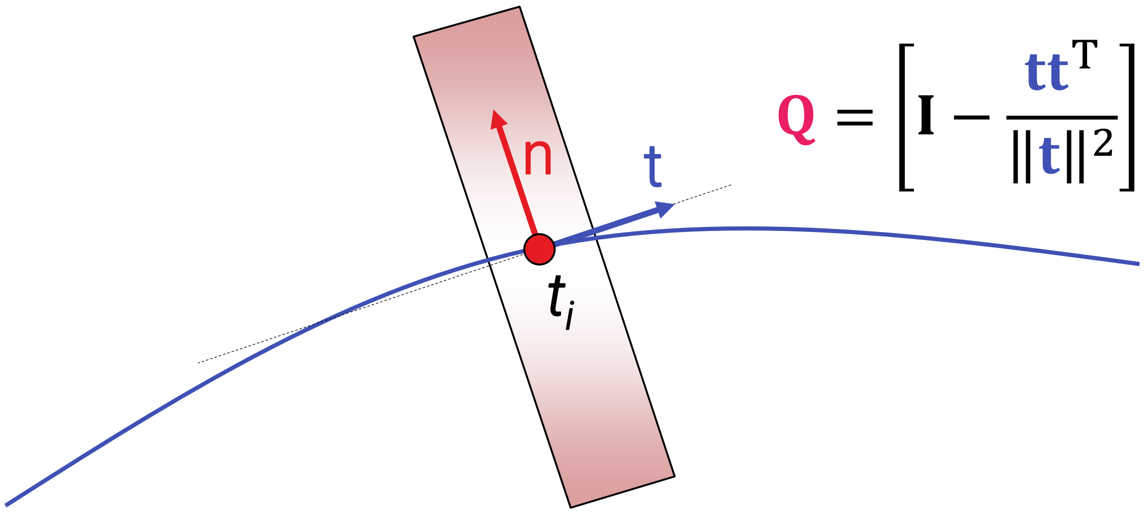
Extension

- Error quadrics
  - Specify the accuracy by error quadrics $\mathbf{Q}_1, \dots, \mathbf{Q}_n$:

$$E(f) = \int_{t=t_1}^{t=t_n} \left[\frac{d^2}{dt^2} f(t)\right]^2 dt + \lambda \sum_{i=1}^{n} (f(t_i) - \mathbf{p}_i)^2$$

$$E(f) = \int_{t=t_1}^{t=t_n} \left[\frac{d^2}{dt^2} f(t)\right]^2 dt + \lambda \sum_{i=1}^{n} (f(t_i) - \mathbf{p}_i)^{\mathrm{T}} \mathbf{Q}_i (f(t_i) - \mathbf{p}_i)$$

Rank-Deficient Quadrics



$$\mathbf{Q} = \left[\mathbf{I} - \frac{\mathbf{t}\mathbf{t}^{\mathrm{T}}}{\|\mathbf{t}\|^2}\right]$$

(linear if $\mathbf{t}$ is fixed a priori)

Error quadric example:

- Permit tangential movement
  - Up to first order
  - Parameter values might be inaccurate
- Rank-$(d-1)$ matrix constraints
- Point-to-normal constraints

Numerical Treatment

Numerical computation

- No closed form solution
- Numerical solution
  - Discretize (finite dimensional function space)
  - Solve for coefficients (coordinate vectors in function space)

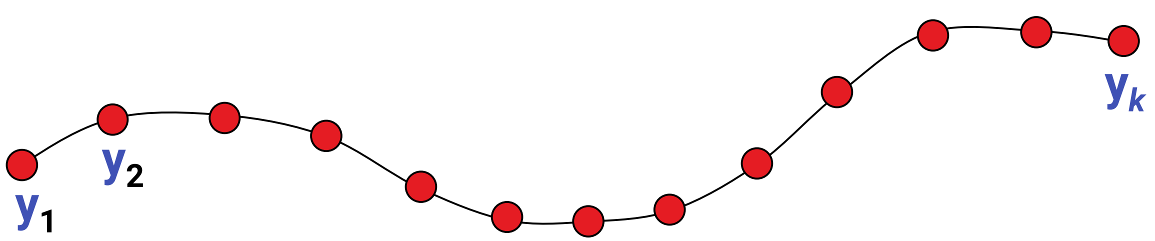
Finite Differences

FD solution:

- Represent curve as array of k values:

t	0	0.1	0.2	...	7.4	7.5
$\mathbf{y}$	$\mathbf{y}_0$	$\mathbf{y}_1$	$\mathbf{y}_2$	...	$\mathbf{Y}_{74}$	$\mathbf{y}_{75}$

- Unknowns are the curve points $y_1, \ldots, y_k$



Discretized Energy Function

- Energy: squared linear expression
  - Quadratic objective function
- Solution by linear system

$$E(f) = \int_{t=t_1}^{t_n} \left[\frac{d^2}{dt^2} \mathbf{f}(t) \right]^2 dt + \sum_{i=1}^{n} \left(\mathbf{f}(t_i) - \mathbf{p}_i \right)^{\mathrm{T}} \mathbf{Q}_i \left(\mathbf{f}(t_i) - \mathbf{p}_i \right)$$

$$E^{(discr)}(f) = \sum_{i=1}^{k} \left[\frac{\mathbf{y}_{i-1} - 2\mathbf{y}_i + \mathbf{y}_{i+1}}{h^2} \right]^2 + \sum_{i=1}^{n} \left(\mathbf{y}_{index(t_i)} - \mathbf{p}_i \right)^{\mathrm{T}} \mathbf{Q}_i \left(\mathbf{y}_{index(t_i)} - \mathbf{p}_i \right)$$

(neglected here: handling boundary values)

Summary

Summary

- Variational approaches look like this:

Optimization: compute $\arg\min E(f)$

Objective: $E(f) = E^{(\text{data})}(f) + E^{(\text{regularizer})}(f)$

Hard constraints: $f \in \mathcal{F} := \{f \mid f \text{ satisfies hard constraints}\}$

- Connection to statistics
  - Bayesian maximum a posteriori estimation
  - $E^{(\text{data})}$ is the data likelihood (log space)
  - $E^{(\text{regularizer})}$ is a prior distribution (log space)

Variational Toolbox

Data Fitting, Regularizer Functionals, Discretizations

Toolbox

In the following:

- We will discuss...
  - ...useful standard functionals.
  - ...how to model soft constraints.
  - ...how to model hard constraints.
  - ...how to discretize the model.
- *Click* & *snap* your custom variational model
  - (Click & snap: add together to a composite energy)

Functionals

Functionals

Standard Functional #1: Function norm

- Given a function

$$f : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$$

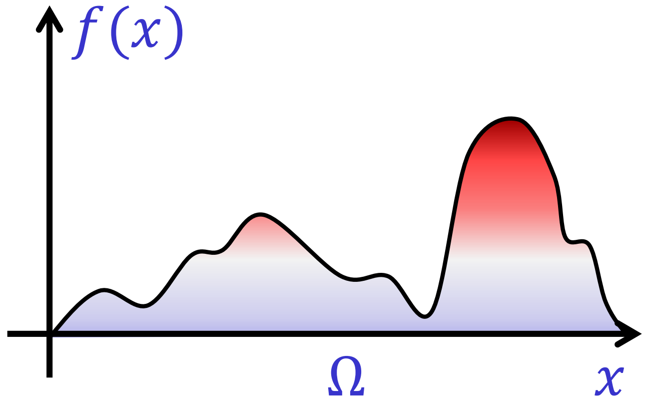
- Minimize

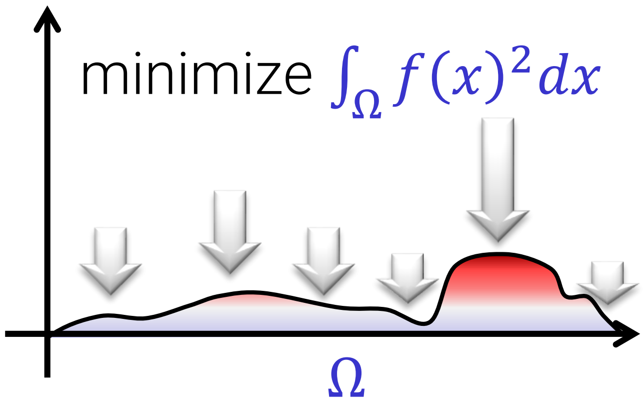
$$E(f) = \|f\|^2 = \int_{\Omega} f(\mathbf{x})^2 d\mathbf{x}$$

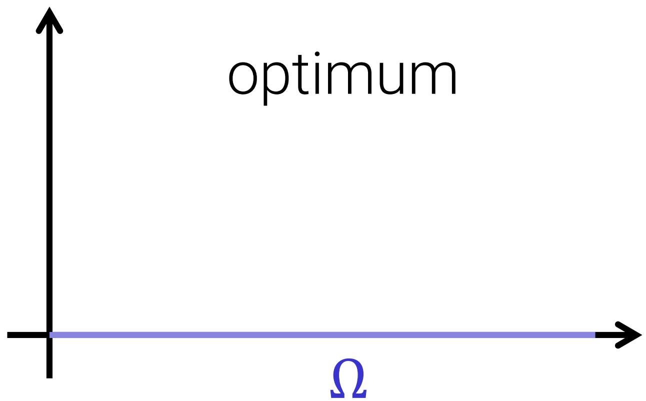
Objective

- Function values should not become too large
- Often useful to avoid numerical problems
    - Positive quadratic energy, then add $\lambda E^{(zero)}$
      $\Rightarrow$ smallest eigenvalue bounded by λ
    - System always solvable

Illustration







Functionals

Standard Functional #2: Harmonic energy

- Given a function

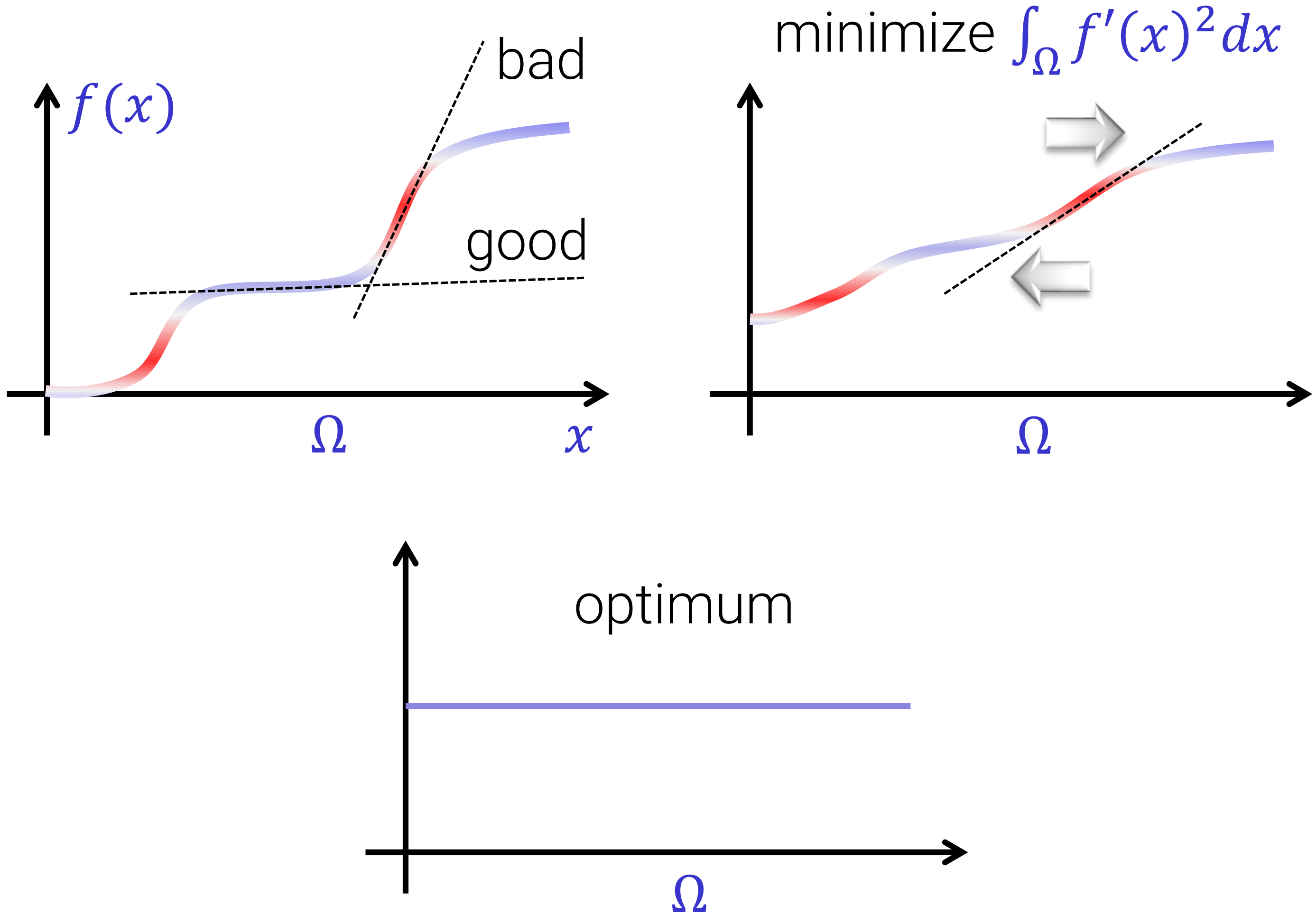
$$f: \Omega \rightarrow \mathbb{R}^n, \Omega \subset \mathbb{R}^m$$

- Minimize:

$$E(f) = \|\nabla f\|^2 = \int_\Omega \left(\nabla f(\mathbf{x})\right)^2 d\mathbf{x}$$

- Minimize differences to neighboring points
- Appears frequently in physics & engineering

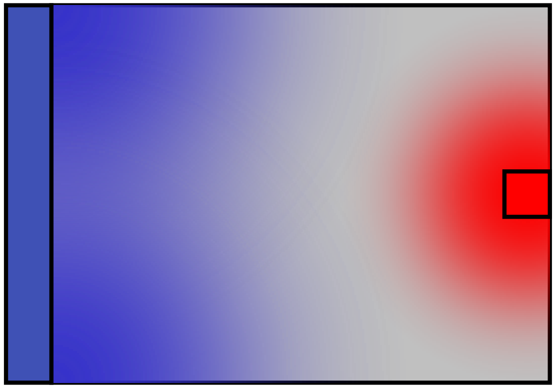
Illustration: 1st Derivatives



Harmonic Energy

Example: Heat equation

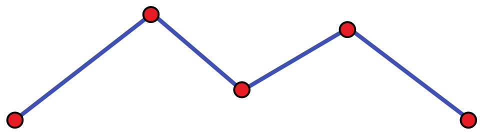
- Metal plate
- Hard constraints:
  - Heat source
  - Heat sink
- Final heat distribution?
  - Heat flow tends to equalize temperature.
    - Stronger heat flow for larger temperature gradients.
  - Gradients become as small as possible.



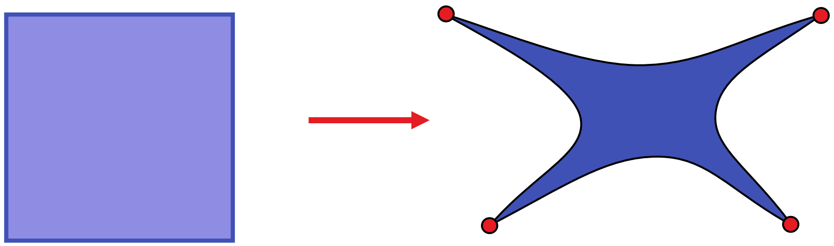
Harmonic Energy

Geometric Effect

- Curves that minimize the harmonic energy
  - Shortest path, a.k.a. polygons

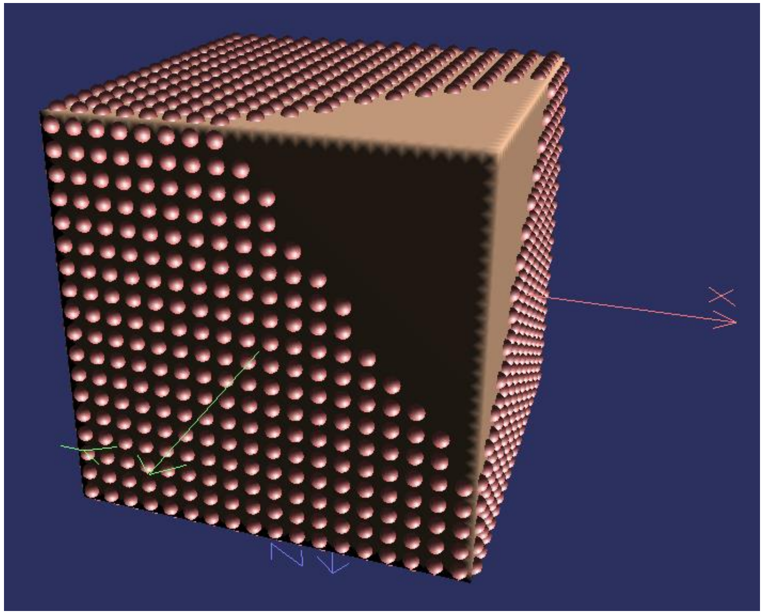


- Two-dimensional parametric surface

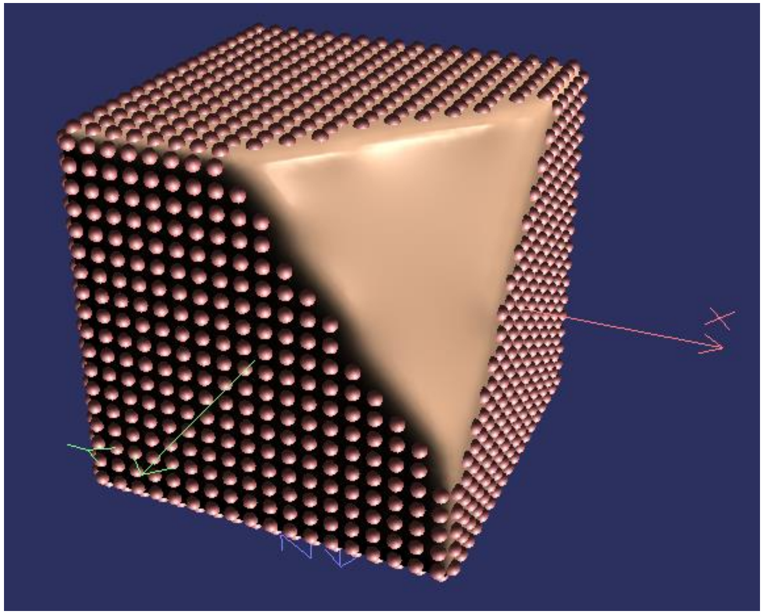


Surface Example

Surface fitting with Laplacian Regularizer



initialization



result

Data attraction: point-to-plane, Gaussian window
Regularizer: minimize triangle edge length

Functionals

Standard Functional #3: Thin plate spline energy

- Given a function

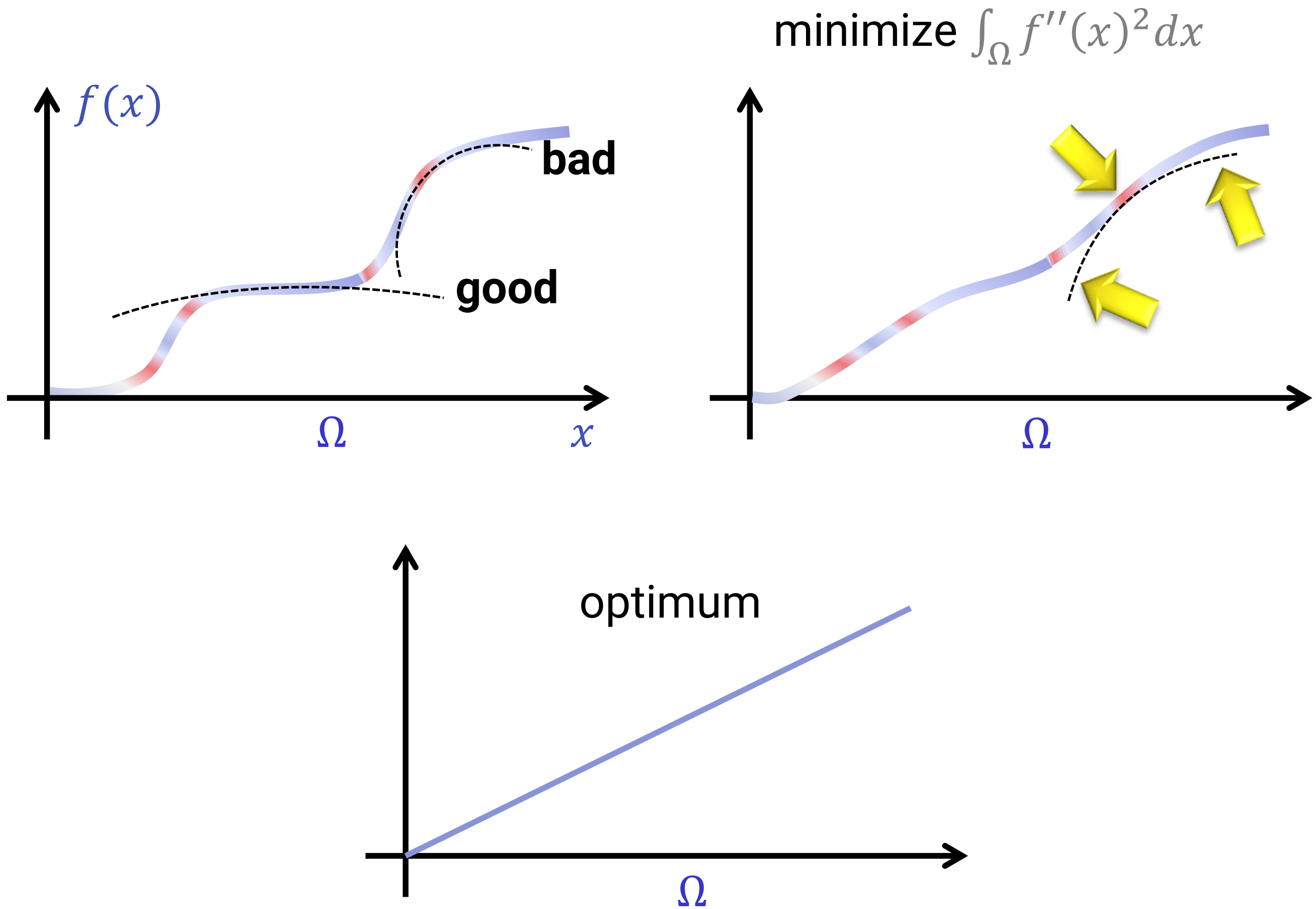
$$f : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$$

- Minimize:

$$E(f) = \int_\Omega \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} f(\mathbf{x}) \right\|^2 d\mathbf{x}$$

- Minimize integral second derivatives (approx. curvature)
  - Yields smooth, low curvature curves & surfaces
  - Exact curvature based energy is non-quadratic
    - Rare in practice

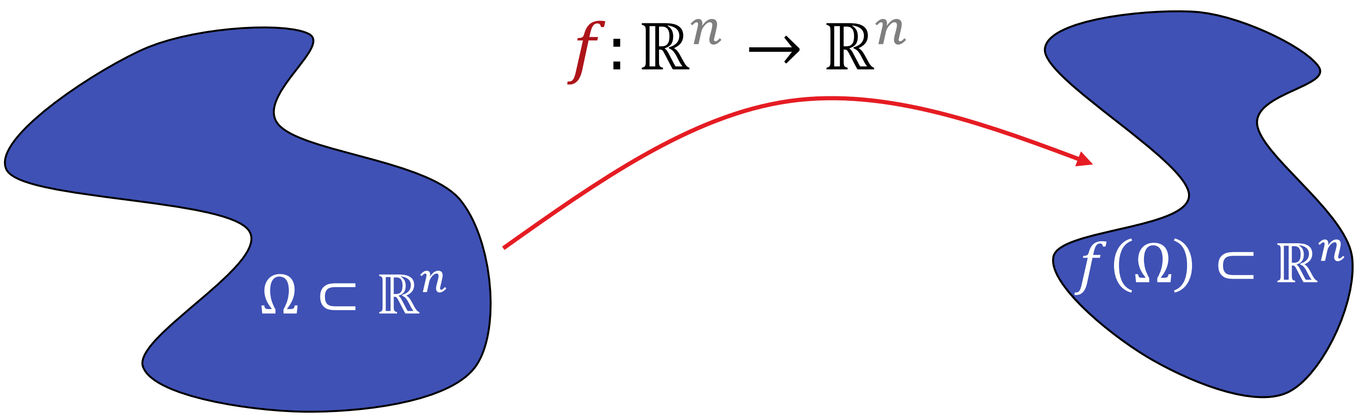
Illustration: 2nd Derivatives



Energies for Vector Fields

Vector fields:

- Now consider volume deformations: $\mathbb{R}^n \to \mathbb{R}^n$
- Object moving over time:
  - $f(\mathbf{x})$ describes its deformation.
  - $f(\mathbf{x}, t)$ describes its motion over time.



Functionals

Standard Functional #4: Green's deformation tensor

> **Remark:** Frobenius Norm
> $$\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|_F^2 = a^2 + b^2 + c^2 + d^2$$

- Given a function

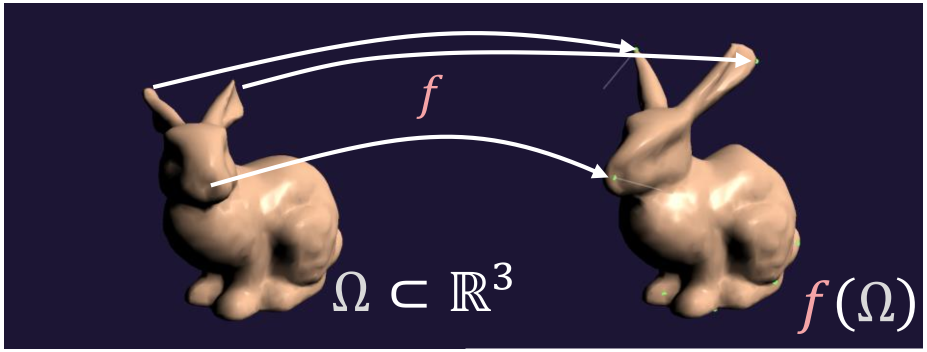
$$f: \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^n$$

- Minimize

$$E(f) = \int_\Omega \left\| \left[[\nabla f(\mathbf{x})]^{\mathrm{T}} [\nabla f(\mathbf{x})] - \mathbf{I} \right] \right\|_F^2 d\mathbf{x}$$

- Physically-based deformation modeling
  - Minimize "metric distortion"
  - Jacobian ∇f is orthogonal $\Leftrightarrow$ $\nabla f \cdot \nabla f^{\mathrm{T}} = \mathbf{I}$
- Invariant under rigid transformations.
  - Bending, scaling, shearing is penalized.
  - Energy is non-quadratic (4-th order).

Green Tensor / Solid Dynamics



Model

- Object $\Omega \subset \mathbb{R}^d$ $(d = 2{,}3)$

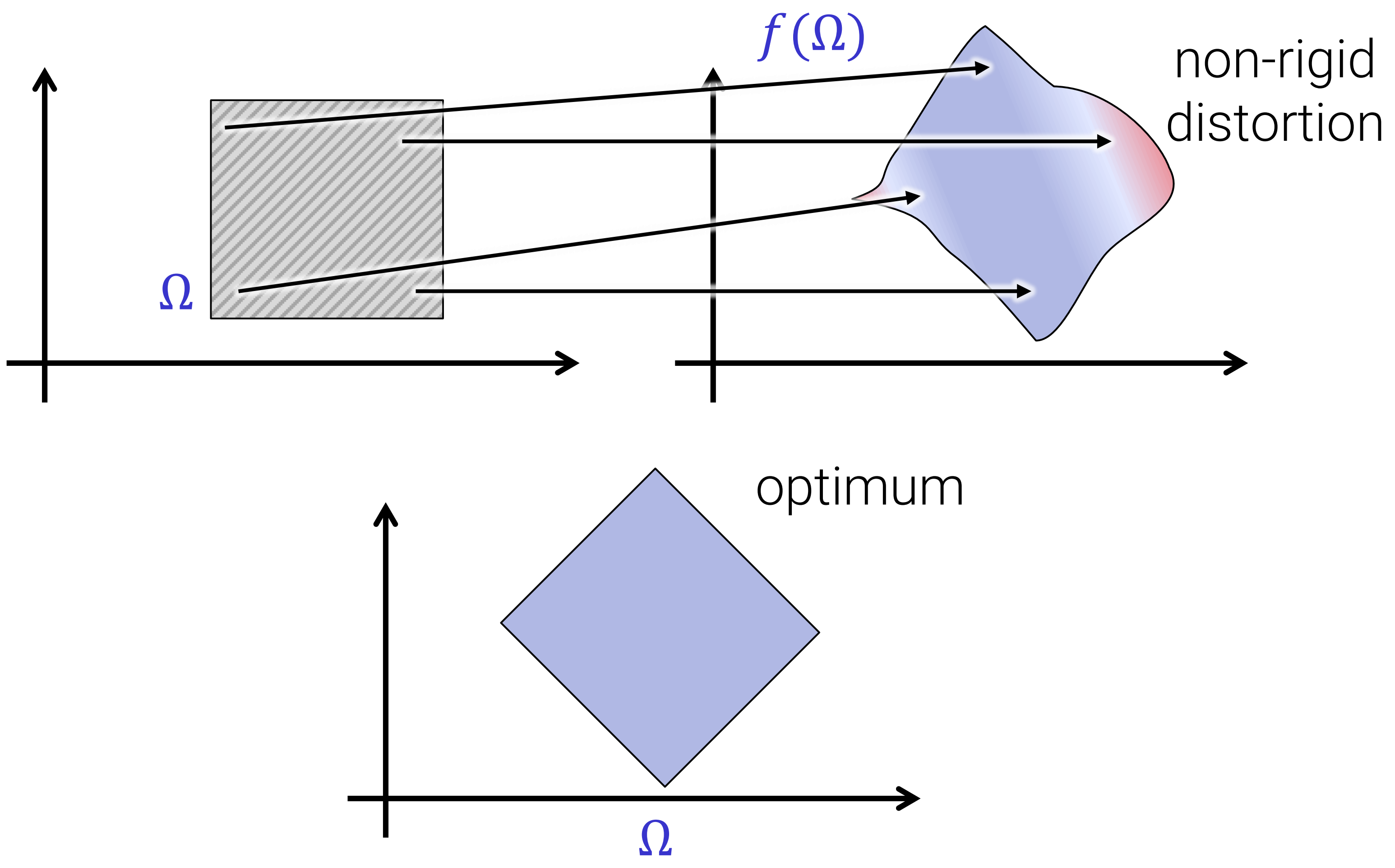
Deformation field $f\colon \Omega \times \mathbb{R} \to \mathbb{R}^n$,

$f(\mathbf{x}, t)$ = new position of point $\mathbf{x}$ at time t

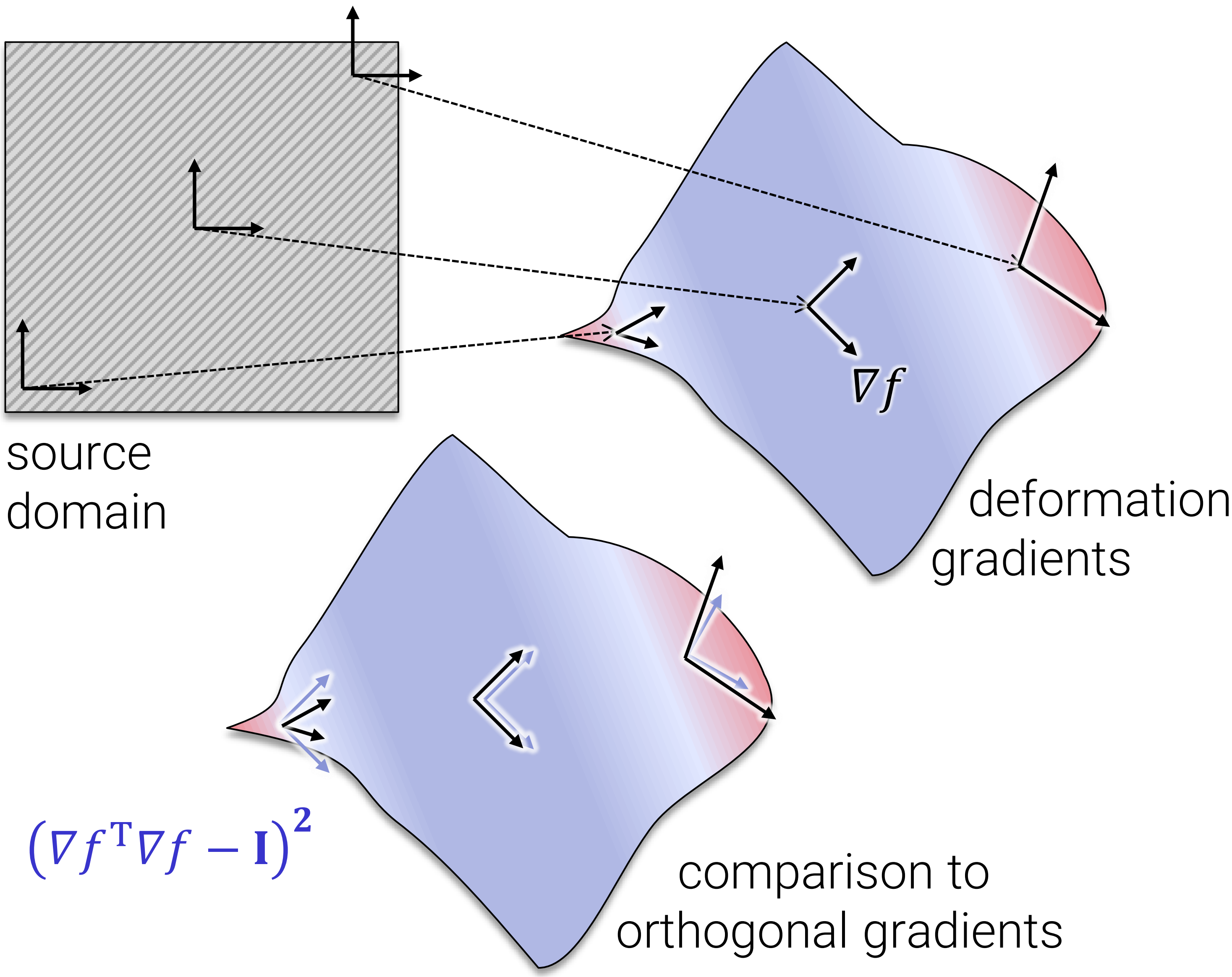
Green Tensor

- (Also) used for modeling deformable solids
- Physically-based deformation modeling
- PDE as Equation of motion

Illustration



Deformation Gradients



Functionals

Standard Functional #5: Volume preservation

- Given a function

$$f : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^n$$

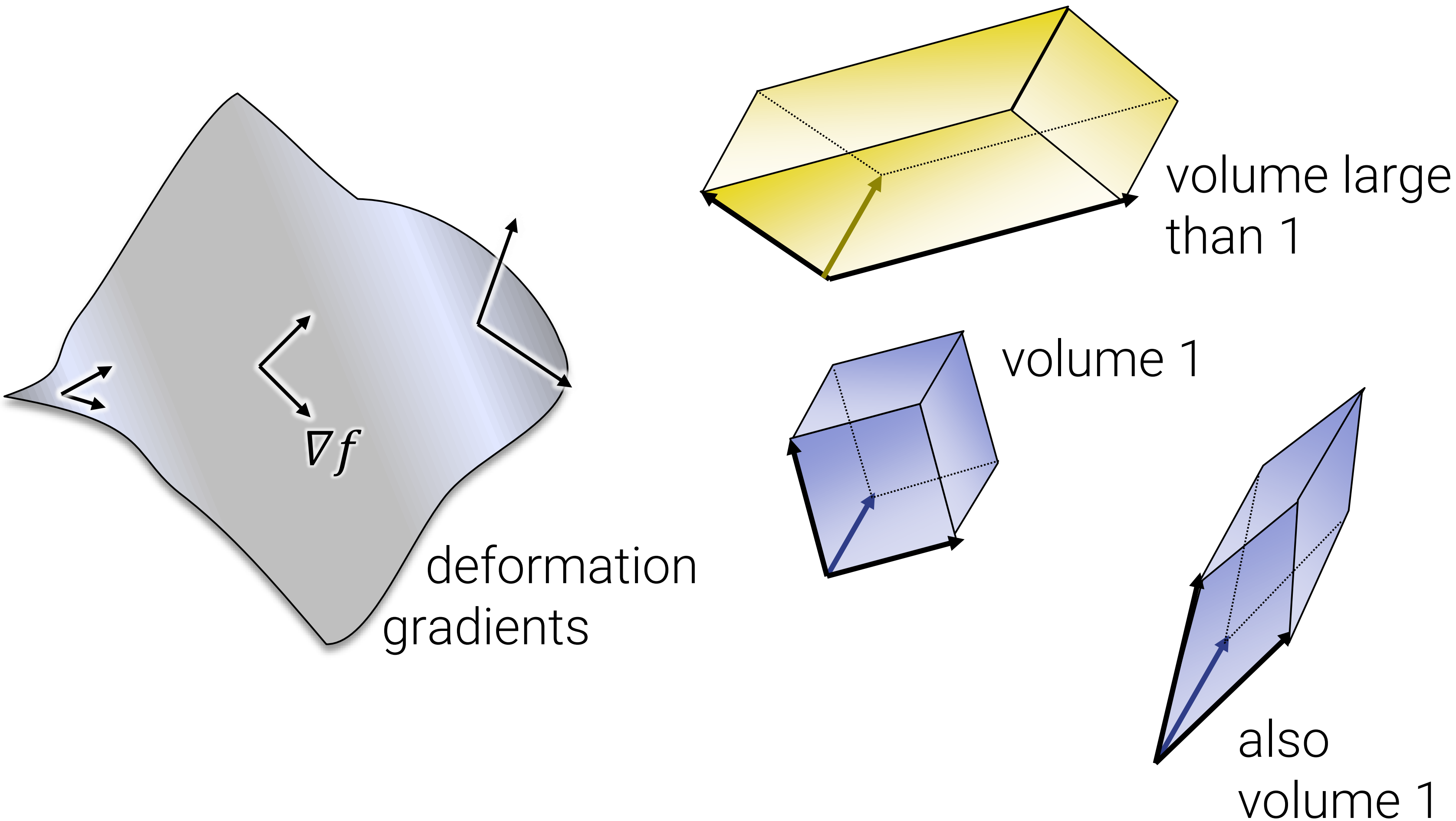
- Minimize

$$E(f) = \int_{\Omega} [\det(\nabla f(\mathbf{x})) - 1]^2 d\mathbf{x}$$

- Objective
  - Minimize *local* volume changes
- Preserve the volume at every point
  - Incompressible materials (for example fluids)
  - Invariant under rigid transformations
  - Non-quadratic (6th-order in 3D)

Illustration

Determinant = area / volume



Functionals

Standard Functional #6: Infinitesimal volume preservation

- Velocity

$$v: \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^n, v(x) = \frac{d}{dt} f(\mathbf{x}, t)$$

- Minimize

$$E(v) = \int_\Omega [\operatorname{div} v(\mathbf{x}, t)]^2 d\mathbf{x}$$

$$= \int_\Omega \left[\frac{\partial}{\partial x_1} v_1(\mathbf{x}, t) + \cdots + \frac{\partial}{\partial x_n} v_n(\mathbf{x}, t) \right]^2 d\mathbf{x}$$

- Minimizes local volume changes in a *velocity field*
- Instantaneous motions
  - Linear, but works only for small time steps
  - Large (rotational) displacements are not covered

Functionals

Standard Functionals #7 & #8: Velocity & acceleration

- Function

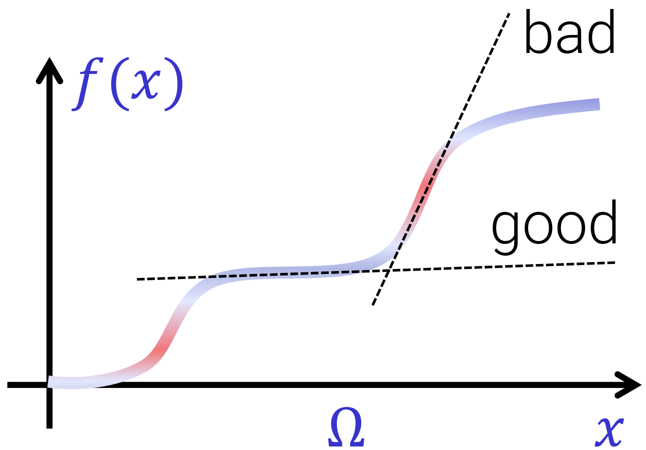
$$f : \Omega \times \mathrm{T} \to \mathbb{R}^n,$$
$$\Omega \subset \mathbb{R}^n, \mathrm{T} = [t_s, t_e] \subset \mathbb{R}$$

- Minimize:

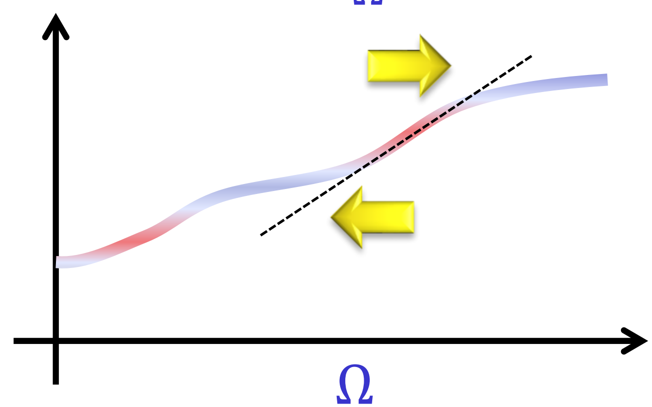
$$E(f) = \iint_{\Omega \times \mathrm{T}} \left(\frac{d}{dt} f(\mathbf{x}, t) \right)^2 d\mathbf{x} dt \qquad E(f) = \iint_{\Omega \times \mathrm{T}} \left(\frac{d^2}{dt^2} f(\mathbf{x}, t) \right)^2 d\mathbf{x} dt$$

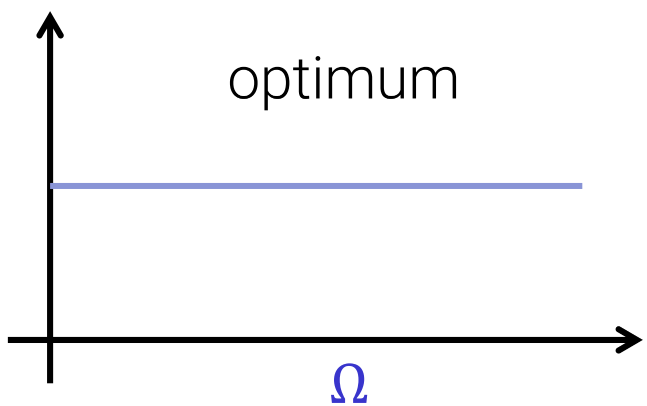
- Objective: minimize velocity / acceleration
  - Air resistance, inertia.

Illustration: 1st Derivatives



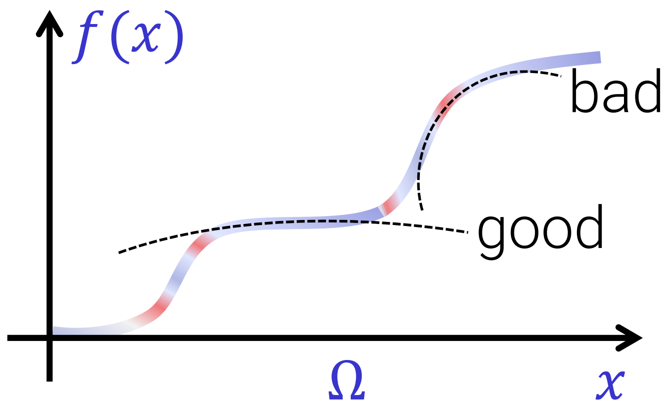
minimize $\int_\Omega f'(x)^2 dx$



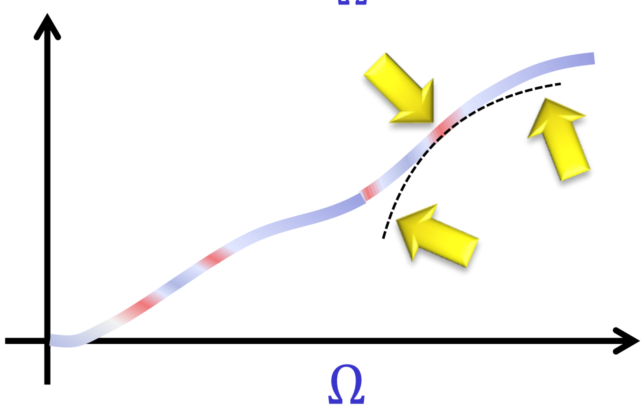


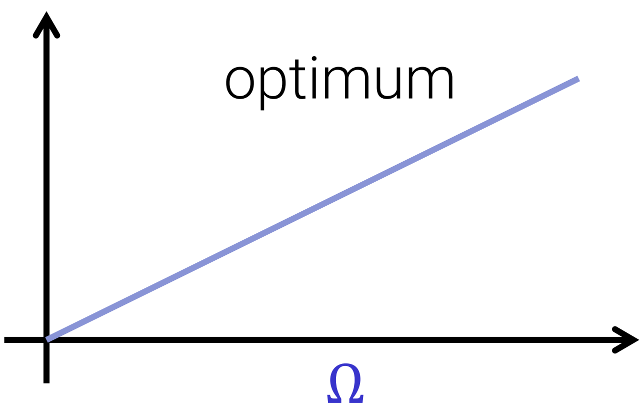
$$E(f) = \iint_{\Omega \times \mathrm{T}} \left(\frac{d}{dt} f(\mathbf{x}, t) \right)^2 d\mathbf{x} dt$$

Illustration: 2nd Derivatives



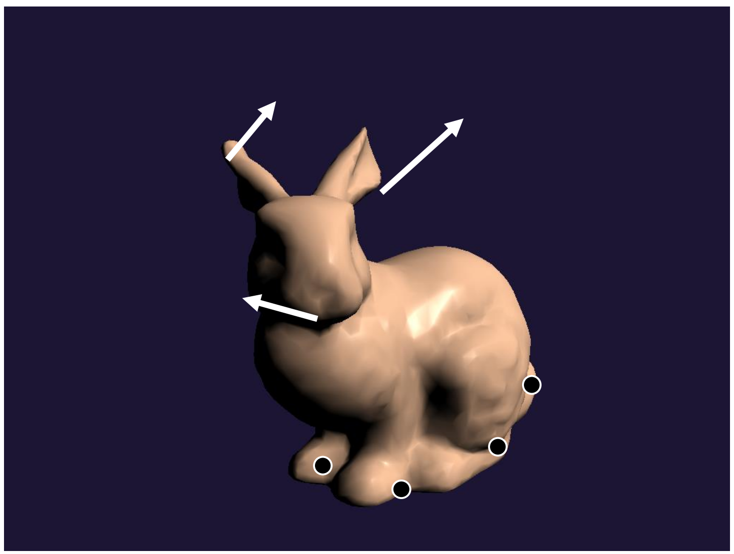
minimize $\int_\Omega f''(x)^2 dx$





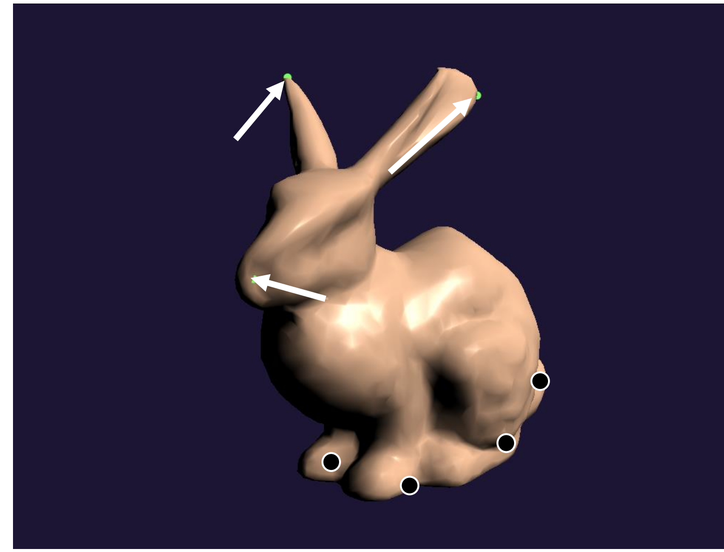
$$E(f) = \iint_{\Omega \times \mathrm{T}} \left(\frac{d^2}{dt^2} f(\mathbf{x}, t) \right)^2 d\mathbf{x}\, dt$$

How does the deformation look like?

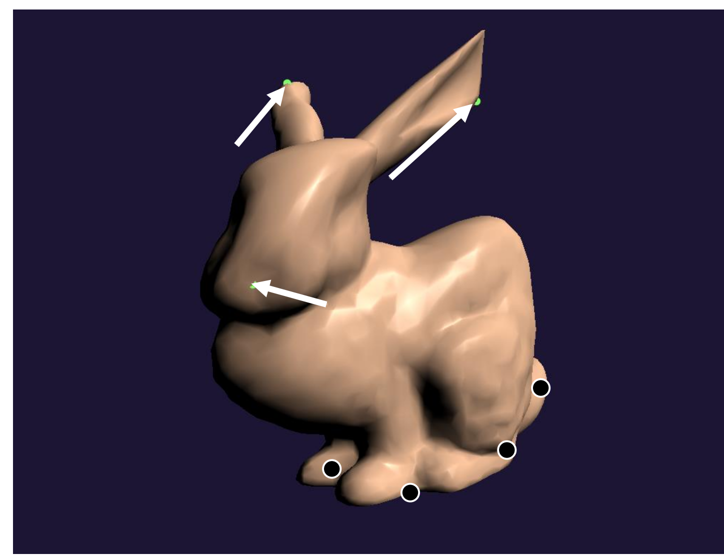


original

as-rigid-as possible volume



thin plate splines



Soft Constraints

Soft Constraints

Penalty functions

- Uniform
- General quadrics
- Differential constraints

Types of soft constraints

- Point-wise constraints
- Line / area constraints

Constraint functions

- Least-squares
- M-estimators

Uniform Soft Constraints

Uniform, point-wise soft constraints:

- Given a function

$$f: \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$$

- Minimize: $E^{(constr)}(\mathbf{f}) = \sum_{i=1}^{n} q_i \left(\mathbf{f}(\mathbf{x}_i) - \mathbf{y}_i\right)^2$

constraint weights (certainty)

prescribed values $(x,y)_i$

Uniform Soft Constraints

General quadratic, point-wise soft constraints:

- Given a function

$$f : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$$

- Minimize:

$$E^{(constr)}(\mathbf{f}) = \sum_{i=1}^{n} \left(\mathbf{f}(\mathbf{x}_i) - \mathbf{y}_i\right)^{\mathrm{T}} \mathbf{Q}_i \left(\mathbf{f}(\mathbf{x}_i) - \mathbf{y}_i\right)$$

constraint weights (general quadratic form, non-negative)

prescribed values $(x,y)_i$

Uniform Soft Constraints

Differential constraints:

- Given a function $f : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$
- Minimize:

$$E^{(constr)}(\mathbf{f}) = \sum_{i=1}^{n} \left(D\mathbf{f}(\mathbf{x}_i) - (D\mathbf{y})_i \right)^{\mathrm{T}} \mathbf{Q}_i \left(D\mathbf{f}(\mathbf{x}_i) - (D\mathbf{y})_i \right)$$

constraint weights (general quadratic form, non-negative)

prescribed values (x,*D*y)$_i$

Differential operator: $D = \begin{pmatrix} \frac{\partial}{\partial x_{i_{1,1}} \ldots \partial x_{i_{k_1,1}}} \\ \vdots \\ \frac{\partial}{\partial x_{i_{1,m}} \ldots \partial x_{i_{k_m,m}}} \end{pmatrix}$

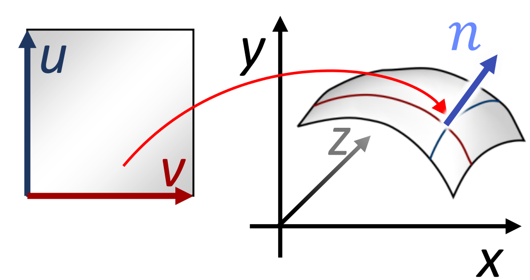
This are still quadratic constraints ($\to$ linear system).

Examples

Examples of differential constraints:

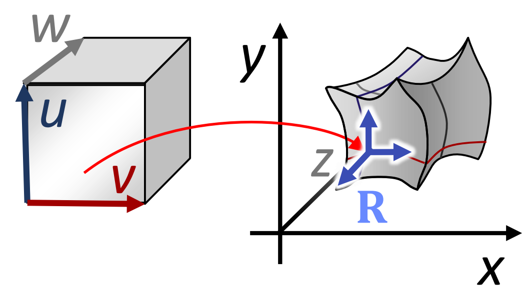
- Prescribe normal orientation of a parametric surface

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \qquad E(f) = \left(\begin{pmatrix} -\partial_u \\ -\partial_v \\ 1 \end{pmatrix} f(u, v) - n \right)^2$$



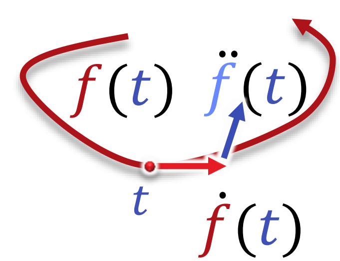
- Prescribe rotation of a deformation field

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \qquad E(f) = \|\nabla f(\mathbf{x}) - \mathbf{R}\|_F^2$$



- Prescribe acceleration of a particle

$$f: \mathbb{R} \rightarrow \mathbb{R}^3, \qquad f(t) = \text{pos.}, \dot{f}(t) = \text{velocity},$$
$$E(f) = \|\ddot{f}(t) - a(t)\|_F^2$$



Line / Area Soft Constraints

Line and area constraints:

- Given a function $f: \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^m$
- Minimize:

$$E^{(constr)}(\mathbf{f}) = \int_{A \subseteq \Omega} \left(\mathbf{f}(\mathbf{x}) - \mathbf{y}(\mathbf{x})\right)^{\mathrm{T}} \mathbf{Q}(\mathbf{x}) \left(\mathbf{f}(\mathbf{x}) - \mathbf{y}(\mathbf{x})\right)$$

quadric error weights (may be position dependent)

prescribed values y(x) (function of position x)

area $A \subseteq \Omega$ on which the constraint is placed (line, area, volume...)

- A.k.a: “transfinite constraints”

Constraint Functions

Typical: quadratic constraints

- $E(x) = f(x)^2$
- Easy to optimize
  - Linear system
- Well-defined critical point
  - Gradient vanishes
- However: sensitive to outliers

Constraint Functions

Alternatives for bad data

- L_1-norm constraints ($E(x) = |f(x)|$)
  - more robust
  - still convex, i.e. can be optimized
- Truncated constraints
  - even more robust
  - non-convex, might be difficult to optimize

Discretization

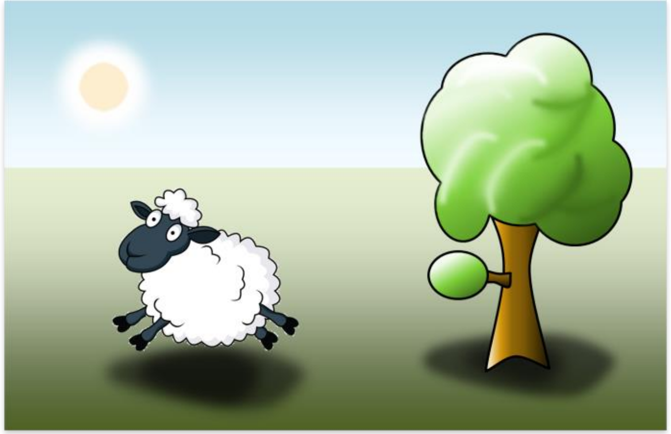
Two Approaches

Finite Differences

- Use grid
  - Replace differentials by differences
  - Replace integrals by sums
- See simple example

Finite Elements

- Linear Ansatz



Linear Ansatz

Linear Ansatz

- We use a linear ansatz:

$$f(\mathbf{x}) \approx \tilde{f}(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i b_i(\mathbf{x})$$

- $\tilde{f}$ lives in a finite dimensional subspace
- Coordinates: $\lambda_1 \dots \lambda_n$

<digression>

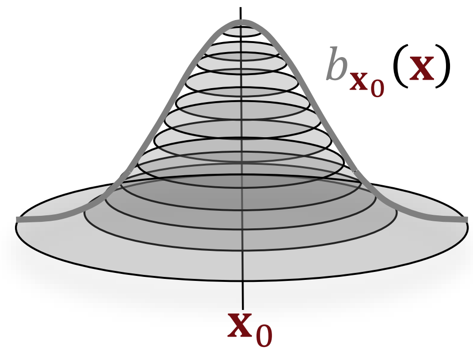
Basis Design?

Which Basis Functions?

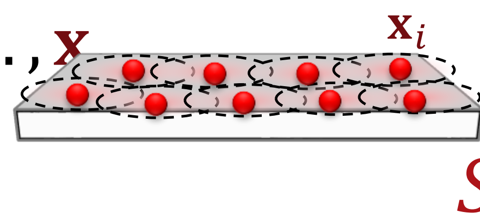
Example

- Radial basis functions (RBFs)

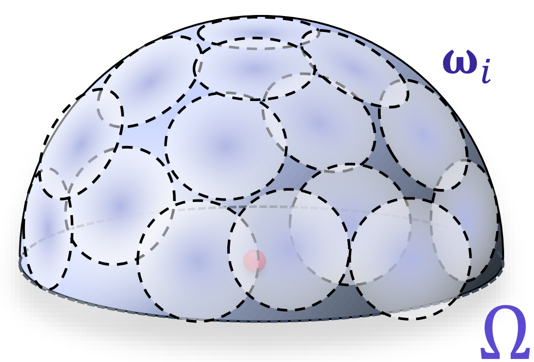
$$b_{\mathbf{x}_0}(x) = \exp\left(-\frac{1}{\sigma^2}(\mathbf{x} - \mathbf{x}_0)^2\right)$$



- Sample surface uniformly with $\mathbf{x}_1, \ldots, \mathbf{x}$



- General domains Ω:
  - Sample uniformly, too
  - Use Euclidean RBFs restricted to Ω



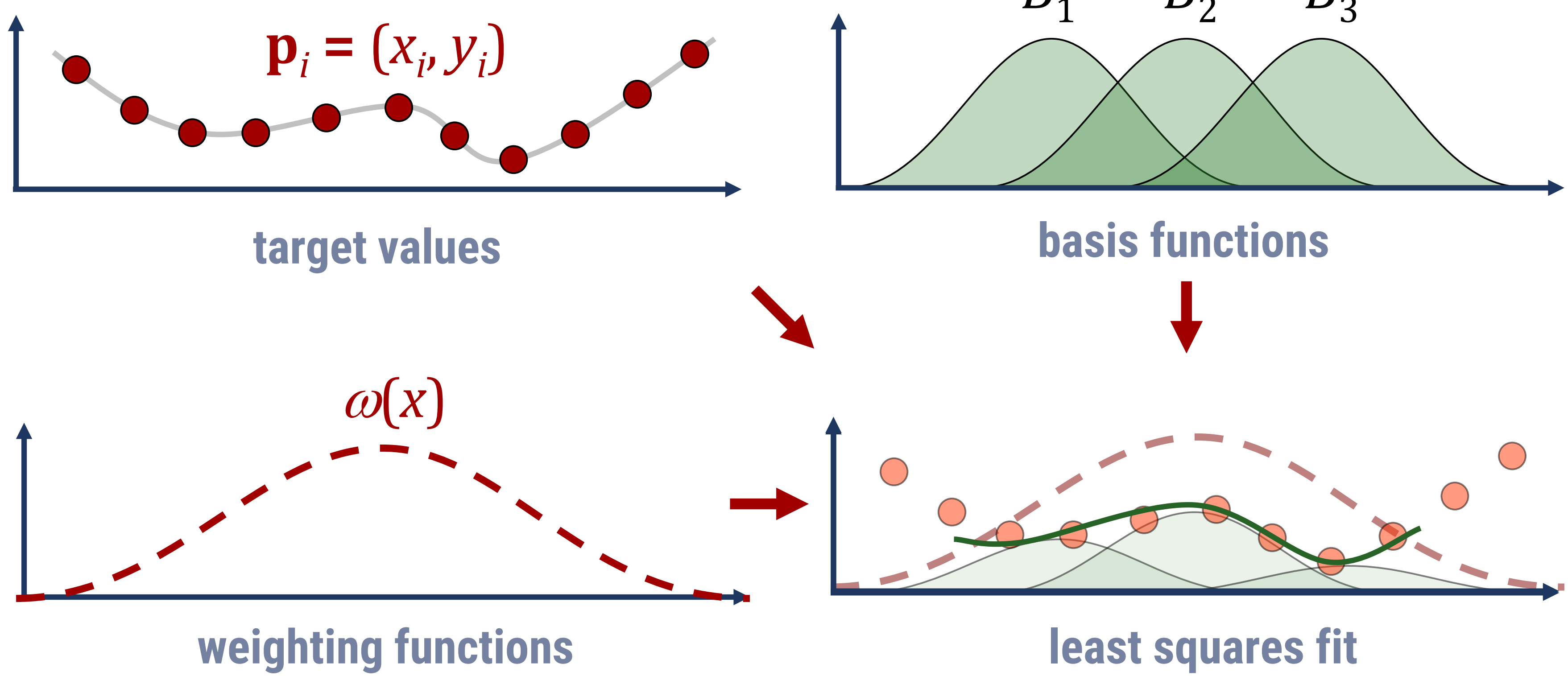
Other bases

Other basis functions

- RBF-like functions with higher consistency order
  - Zero order: Partition of unity
  - First, second, third,… order:
    Polynomial moving least-squares
- Mesh-based FE functions (spline meshes)
- Fourier basis, spherical harmonics, etc.
- Wavelets
  - Finite spatial & frequency support

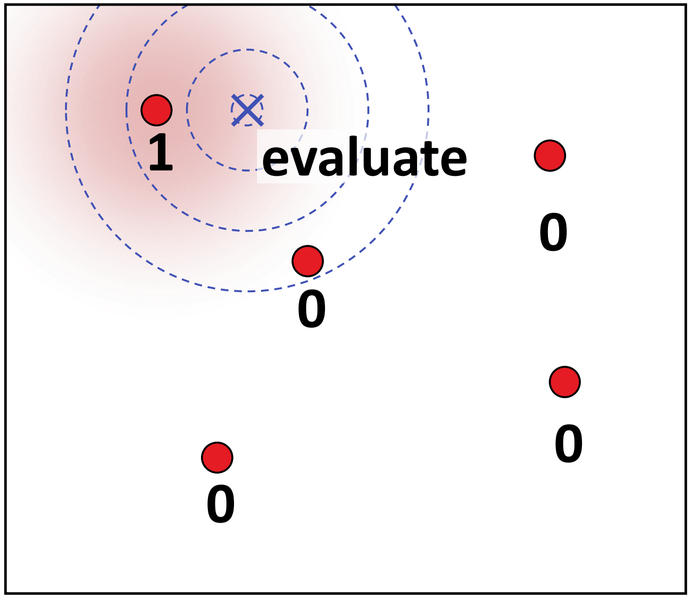
Digression: Meshless MLS-Bases

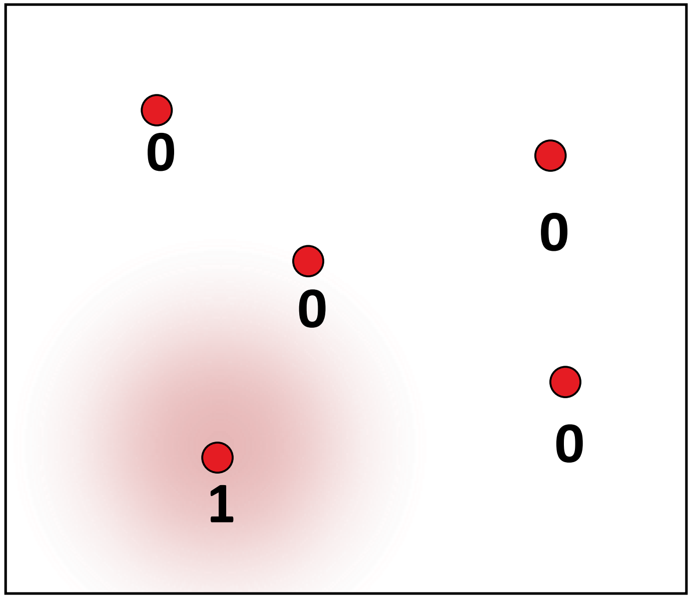
Moving Least Squares



Digression: Meshless MLS-Bases

Constructing the basis





Properties

- Consistency order and smoothness of the MLS-Scheme
- Need to invert matrix to evaluate at each point

</digression>

Back to FE…

Finite Element Discretization

Derive a discrete equation:

- Just plug in the discrete $\tilde{f}$.
- Then minimize the it over the $\boldsymbol{\lambda}$.
- Compute the critical point(s):

$$E\left(\tilde{f}_{\boldsymbol{\lambda}}(\mathbf{x})\right) \to \min. \quad \Rightarrow \quad \forall i = 1, \ldots, k: \frac{\partial}{\partial \lambda_i} E\left(\tilde{f}_{\boldsymbol{\lambda}}(\mathbf{x})\right) = 0$$

Solve Equations

- Quadratic functionals: linear system.
- Non-linear, smooth functionals:
  Newton, Gauss-Newton, L-BFGS, …

Example

(Abstract) example:

- Minimize square integral of a differential operator D
  - Quadratic differential constraints
- Data term: Match points $f(\mathbf{x}_i) = \mathbf{y}_i$
  - Soft constraints
- Yields quadratic optimization problem in the coefficients

Example

(Abstract) example (cont):

$$E(f) = \int_\Omega \left(Df(\mathbf{x})\right)^2 d\mathbf{x} + \mu \sum_{i=1}^{n} (f(\mathbf{x}_i) - \mathbf{y}_i)^2$$

$$E(\tilde{f}_{\boldsymbol{\lambda}}) = \int_\Omega \left(D \sum_{i=1}^{k} \lambda_i b_i(\mathbf{x}) \right)^2 d\mathbf{x} + \mu \underbrace{\sum_{i=1}^{n} \left(\sum_{j=1}^{k} \lambda_j b_j(\mathbf{x}_i) - \mathbf{y}_i \right)^2}_{DataTerm}$$

$$= \int_\Omega \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j [D b_i(\mathbf{x})] \, [D b_j(\mathbf{x})] \, d\mathbf{x} \; + \; \mu \; DataTerm(\boldsymbol{\lambda})$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j \int_\Omega D b_i(\mathbf{x}) D b_j(\mathbf{x}) \, d\mathbf{x} \; + \; \mu \; DataTerm(\boldsymbol{\lambda})$$

Example 1

Image Reconstruction

Image Reconstruction Model

Problem statement

- Measured 2D pixel image
- Distorted by noise
- Want to remove noise

Bayesian problem modeling

- Model of measurement process
- Prior distribution on images (this is Bayesian)

Inference: Maximum-a-posteriori

Model

Image

- $x_{i,j}$ with $i = 1 \dots w, j = 1, \dots, h$
- continuous model: $f: [1, w] \times [1, h] \rightarrow \mathbb{R}$

Probability space

- $\Omega = \mathbb{R}^{w \times h}$
- Probability measure on sigma-algebra on $\mathbb{R}^{w \times h}$
- Continuous model "f": mathematically very involved
  - We restrict ourselves to finite-dimensional probabilistic models

Model

Bayes rule

$$P(X|D) \sim P(D|X) \cdot P(X)$$

Likelihood

- $P(D|X) = \prod_{i=1}^{w} \prod_{j=1}^{h} P(d_i|x_i)$ (i.i.d. noise)

  $= \prod_{i=1}^{w} \prod_{j=1}^{h} \mathcal{N}_{d_i,\sigma_D}(x_i)$ (Gaussian noise)

  $= \prod_{i=1}^{w} \prod_{j=1}^{h} \left[\frac{1}{\sigma_D\sqrt{2\pi}} e^{-\frac{(x_i-d_i)^2}{2\sigma_D^2}}\right]$

  (Gaussian distribution)

Model

Likelihood

- $P(D|X) = \prod_{i=1}^{w} \prod_{j=1}^{h} \left[\frac{1}{\sigma_D \sqrt{2\pi}} e^{-\frac{(x_i - d_i)^2}{2\sigma_D^2}} \right]$

Neg-Log-Likelihood

$$E(D|X) := -\ln P(D|X) = \sum_{i=1}^{w} \sum_{j=1}^{h} \frac{(x_i - d_i)^2}{2\sigma_D^2} + \frac{wh}{\sigma_D \sqrt{2\pi}}$$

independent of x_i

Model

Prior

- Assumption: Large image gradients are unlikely
- Gaussian distribution on Gradients
- Neg-log-likelihood: $\frac{1}{2\sigma^2} \|\nabla f\|^2$
- Discreet:

$$\mathrm{E}(X) \coloneqq -\ln P(X) = \sum_{i=1}^{w-1} \sum_{j=1}^{h-1} \frac{\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(x_{i,j+1} - x_{i,j}\right)^2}{2\sigma_X^2} + \frac{wh}{\sigma_X \sqrt{2\pi}}$$

independent of x_i

Minimization Problem

Minimize

$$E(D|X) + \mathrm{E}(X)$$

$$= \sum_{i=1}^{w}\sum_{j=1}^{h}\frac{(x_i - d_i)^2}{2\sigma_D^2} + \sum_{i=1}^{w-1}\sum_{j=1}^{h-1}\frac{\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(x_{i,j+1} - x_{i,j}\right)^2}{2\sigma_X^2}$$

Equivalent minimization objective

$$\sum_{i=1}^{w}\sum_{j=1}^{h}(x_i - d_i)^2 + \frac{\sigma_X^2}{\sigma_D^2}\sum_{i=1}^{w-1}\sum_{j=1}^{h-1}\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(x_{i,j+1} - x_{i,j}\right)^2$$

Continuous

$$\int_\Omega \left(f(\mathbf{x}) - d(\mathbf{x})\right)^2 d\mathbf{x} + \frac{\sigma_X^2}{\sigma_D^2}\int_\Omega \|\nabla f(\mathbf{x})\|^2 d\mathbf{x}$$

Modeling I

Looks familiar?

- This is the same objective as in the modeling I assignment (sheet 06).
- Solution via linear system

Variant

- Penalize l_1 norm instead of l_2 norm of gradients

$$\int_\Omega \left(f(\mathbf{x}) - d(\mathbf{x})\right)^2 d\mathbf{x} + \frac{\sigma_X^2}{\sigma_D^2} \int_\Omega \|\nabla f(\mathbf{x})\|^1 d\mathbf{x}$$

- Laplace distribution (double exponential)
- Yields sharper images (natural image statistics)

Technical Remark

Image Prior

$$-\ln P(X) = \sum_{i=1}^{w-1} \sum_{j=1}^{h-1} \frac{\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(x_{i,j+1} - x_{i,j}\right)^2}{2\sigma_X^2} + \frac{wh}{\sigma_X \sqrt{2\pi}}$$

- This is an "improper prior"
  - Does not integrate to one!
  - Infinite subspaces without penalty
- Formal fix
  - Assume broader prior on function value itself: $f \sim N_{0,\sigma_{very\ large}}$
- For MAP estimation, this does not matter
  - We just find a point of maximum density
  - Integration not required


ground truth
reconstruction (Gaussian prior)
input (noisy)
reconstruction (Laplace prior)